

Optimal Reliability in Design for Fatigue Life

Part I – Existence of Optimal Shapes

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Abstract: Fatigue describes the damage or failure of material under cyclic loading. Activation and deactivation operations of technical units are important examples in engineering where fatigue and especially low-cycle fatigue (LCF) play an essential role. A significant scatter in fatigue life for many materials results in the necessity of advanced probabilistic models for fatigue. Moreover, structural shape optimization is of increasing interest in engineering, where with respect to fatigue the cost functionals are motivated by their predictability for the integrity of the component after a certain number of load cycles. But mathematical properties such as the existence of the shape derivatives are desirable, too. Deterministic design philosophies that derive a predicted component life from the average life of the most loaded point on the component plus a safety factor accounting for the scatter band do not have this favorable property, as taking maxima is not a differentiable operation. In this work we present a new local probabilistic model for LCF. This model constitutes a new link between reliability statistics, shape optimization and structural analysis which considers the perspective of fatigue but also fits into the mathematical setting of shape optimization. The cost functionals derived in this way are too singular to be H^1 lower semi-continuous. We therefore have to modify the existence proof of optimal shapes [20] for the case of sufficiently smooth shapes using elliptic regularity, uniform Schauder estimates and compactness of certain subsets in $C^k(\Omega^{\text{ext}}, \mathbb{R})$ via the Arzela-Ascoli theorem, where Ω^{ext} is some shape containing all admissible shapes. Here, we analyze a class of cost functionals which consists of volume and surface integrals whose integrands include derivatives of the displacement field up to second order. Moreover, we extend our existence results to high-cycle fatigue (HCF) and deterministic models of fatigue.

1 Introduction

A design being made from material will fail if the material degradation due to loading exceeds certain limits. Reliability, i.e. the absence of failure, is thus the ultimate goal of structural design. Whether a design will operate safely under certain load conditions depends on the failure mechanisms which are very diverse for different material classes. Degradation can occur as a function of operating time, which is e.g. the case for creep damage. Or it can occur

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when small plastic deformations under cyclic loading pile up and result in a crack. This is called fatigue which can be differentiated into high-cycle fatigue (HCF) and low-cycle fatigue (LCF). In this work we focus on LCF in conjunction with polycrystalline metals and consider the number of load cycles until crack initiation.

The LCF failure mechanism – confer [6], [30], [15] and [36] – with respect to polycrystalline metal leads to the stochastic nature of LCF which prevails on the macroscopic scale, where a statistical scatter of a factor 10 between the highest and lowest load cycles to crack initiation is a common phenomenon, even under lab conditions. This makes it obvious that the reliability of a design has to be understood quantitatively as the probability of the absence of failure after a given number of loads. Thus reliability statistics, see e.g. [13], should play a vital part in LCF design. Furthermore, LCF cracks initiate at the surface and they are small in the initiation phase. Therefore, they will only influence stress fields on micro- and mesoscales and so it is reasonable to assume that crack formation in one region of the component surface is not influenced by the crack forming process on another part. This forces us to see crack formation as a problem of spatial statistics [34]. We thus infer that hazard rates for crack initiation have to be integrals over some local function depending on the local stress or strain fields. Therefore, hazard rates for the component can be expressed by a surface integral over some crack formation intensity function depending on local fields. The latter can also be seen as the density for the intensity measure of a poisson point process (PPP) on $[0, \infty) \times \partial\Omega$, confer [25] and [5]. Here, the first component stands for ‘time’ measured in cycles. The very interesting article [15] also emphasizes the role of the PPP.

In this work we model hazard rates and intensity densities of the PPP with a rather conservative approach. That is, we assume that the intensity measure is of scale-shape type, as it is the case for many distributions in reliability statistics. Furthermore, we assume that the scale variable N_{det} is of the same functional form as the usual Coffin-Manson-Basquin equation, confer [6] and (5) below. Although we are not aware of any empirical study showing that LCF failure time distributions are Weibull, we determine the shape of the intensity measure according to this approach for mathematical simplicity. It then follows that the number N of cycles of first crack initiation will be a Weibull distribution as well. The Weibull scale parameter is given by the inverse of the $L^m(\partial\Omega, \mathbb{R})$ -norm of the inverse of variable N_{det} . In contrast to [15] this purely phenomenological approach avoids detailed modeling at the meso scale which facilitates calibration with experiments. Both approaches have the use of Weibull distributions in common. From a materials engineering point of view our new model has the significant advantage – compared to standard methods in fatigue – of bypassing the standard specimen approach and of considering size effects.

As LCF is driven by cyclic loads, the reaction of the component to these loads needs to be taken into account as well. Usually, this is done via a continuum mechanics approach and here we do follow. Furthermore, we restrict to linear elasticity as the fundamental equation that determines the loads. This seems to be in contradiction with the need to model also plastic deformation to capture the LCF damage mechanism. But there are time honored procedures that bypass this problem, namely elastic-plastic strain conversion of Neuber and Glinka, see [26]. Although those seems not quite up to date, they are still often used in engineering and they have considerable mathematical advantages when it comes to shape optimization – and this will be the next topic.

So far we have considered the component shape Ω as fixed. It is the task of design teams, to choose the shape such that reliability, performance and cost are in an optimal equilibrium.

Here, we consider the disciplinary task of structural design alone and hence focus on reliability. Narrowing in even further, reliability is understood as the probability of survival (no crack initiation) with respect to LCF at a given number of cycles. An optimal design with respect to LCF will then be a design, that maximizes, under given load conditions, the survival function $S_N(n) = 1 - F_N(n)$ with F_N the cumulative density function of the random LCF crack initiation time N of the component.

From a mathematical point of view, there are quite a few studies on the question of optimal design, see e.g. [20] and references therein. The problem of optimal design can be seen as a PDE restricted optimization problem with control parameter Ω . Compactness of the set of admissible shapes Ω and lower semi-continuity of cost functionals $J(\Omega, u(\Omega))$ to be minimized are essential ingredients. Here, $u(\Omega)$ uniquely solves the linear elasticity equation on Ω with boundary conditions on $\partial\Omega$. In this context, linear elasticity is treated within the weak formulation [10]. Consider a sequence given by $u_n = u(\Omega_n) \in [H^1(\Omega_n)]^3$ for $n \in \mathbb{N}$. Under a few assumptions on $\partial\Omega_n$ it has been shown that $u_{n_k}^{\text{ext}} \rightarrow u^{\text{ext}}$ strongly in $[H^1(\Omega^{\text{ext}})]^3$ for a subsequence u_{n_k} , whenever $\Omega_n \rightarrow \Omega$ in a suitable sense [20]. Here, Ω^{ext} contains all admissible shapes and u^{ext} denotes a suitable extension of functions to Ω^{ext} which are defined on a corresponding admissible domain.

Unfortunately, in the context of LCF a problem occurs that prohibits the application of these results: The LCF failure mechanism leads to cost functionals $J(\Omega, u)$ – the probability of failure at given cycles with displacement field u – that depend on ∇u and are surface integrals. While the components of ∇u are in $L^2(\Omega)$ and of $u|_{\partial\Omega}$ in $L^2(\partial\Omega)$ for $u \in [H^1(\Omega)]^3$, this does not guarantee that $\nabla u|_{\partial\Omega}$ is a well defined function. A proof of lower semi-continuity for the probability of crack initiation as a functional on $[H^1(\Omega)]^3$ therefore seems to be problematic or even wrong. Therefore, we have to look for an alternative strategy to prove the existence of shapes with optimal reliability for LCF.

We address a general class of cost functionals which considers LCF survival and failure probabilities as defined in Definition 4.1 which are apparently $[C^2(\Omega^{\text{ext}})]^3$ continuous. Although this seems to be a very rough way of modeling again, we impose smoothness conditions on $\partial\Omega$ and on the loads and boundary values in order to obtain strong solutions in $[C^3(\Omega)]^3$ for a domain Ω in the case of a fixed Dirichlet and a variable von Neumann boundary. Note that we even consider third derivatives to be able to extend our models to stress gradients. The key here is uniform elliptic regularity and Schauder estimates [2]. This is however still not enough as we need to prove the existence of a limiting function $u \in [C^3(\Omega^{\text{ext}})]^3$ for a subsequence $u(\Omega_{n_k})$ of solutions $u(\Omega_n)$ of the linear elasticity problem on admissible shapes Ω_n suitably continued to Ω^{ext} , $\Omega_n \subseteq \Omega^{\text{ext}}$. But using the Arzela-Ascoli theorem, revisiting the Schauder estimates and proving that they are uniform with respect to the set of admissible shapes leads to the existence of such a limiting function.

The existence of shapes that are optimally reliable with respect to the probabilistic LCF damage mechanism can thus be proven mathematically in the given set up and HCF failure mechanism can be considered as well. But this is not special for the probabilistic approach. If we apply a deterministic LCF lifing model, we need to minimize peak stress under given loads in order to maximise the LCF life. Peak stress is $C^1(\Omega^{\text{ext}})$ continuous as well and thus we obtained existence of optimal shapes for the deterministic approach to LCF as a by-product. The same can be achieved for HCF. It is special to the probabilistic approach to fatigue that it is described by a cost functional which is a local integral of the displacement field and its derivatives. And this is exactly what is needed in shape sensitivity analysis [35] which we will

address in our work [19] with respect to the probabilistic model. Shape sensitivity analysis in conjunction with the adjoint method is a highly effective procedure to find – after suitable discretization – shapes of (local) optimal reliability via a shape gradient decent method. Confer [32] for an application in a different context.

This paper unifies some aspects of materials engineering, reliability statistics, elliptic PDE and shape optimization. We therefore give some compilations of known results from the respective fields to keep the article self contained. In particular, Sections 2 and 3 repeat some basics from materials engineering. Section 4 motivates our main definition – the cost functional for local probabilistic LCF. Here, HCF in a probabilistic sense and a deterministic model for fatigue are discussed as well. An abstract setting for existence analysis of shape optimization is given in Section 5 where we also address C^k -admissible domains. This abstract setting provides the structure of Section 6 where we derive existence results for shape optimization problems with respect to linear elasticity and to C^k -admissible domains. Here, we focus on a general class of cost functionals of integral form without convexity constraints. At the end of Section 6 we apply our existence results to the probabilistic and deterministic models of Section 4 for LCF and HCF.

2 Linear Isotropic Elasticity and Plasticity

In order to describe the behavior of components from polycrystalline metal under external loading, we employ linear isotropic elasticity which is justified by the assumption of isotropic material behavior at scales significantly larger than the grain size and of sufficiently small deformations. We closely follow [12], [23] and [6], respectively.

Let $\Omega \subset \mathbb{R}^3$ be a domain which represents the component shape filled with a deformable medium such as polycrystalline metal which is initially at equilibrium. Moreover, let ν be the normal of the boundary $\partial\Omega$, let $f : \Omega \rightarrow \mathbb{R}^3$ be an external load and let $u : \Omega \rightarrow \mathbb{R}^3$ be the displacement field. Finally, let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ be a partition where $\partial\Omega_D$ is clamped and on $\partial\Omega_N$ a normal load $g : \partial\Omega_N \rightarrow \mathbb{R}^3$ is imposed. Then, according to [12] the mixed problem of linear isotropic elasticity is described by:

$$\begin{aligned} \nabla \cdot \sigma^e(u) + f &= 0 && \text{in } \Omega, \\ \sigma^e(u) &= \lambda(\nabla \cdot u)I + \mu(\nabla u + \nabla u^T) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega_D, \\ \sigma^e(u) \cdot \nu &= g && \text{on } \partial\Omega_N. \end{aligned} \tag{1}$$

Here, λ and μ are the Lamé coefficients. The linearized strain rate tensor $\varepsilon^e(u) : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ is defined as $\varepsilon^e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, i.e. $\varepsilon_{ij}^e = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ for $i, j = 1, 2, 3$. Approximate numerical solutions can be computed by a finite element approach, confer [23] and [12].

In the following we phenomenologically discuss time-independent plasticity as we cannot assume a completely linear elastic material behavior when talking about fatigue. Plastic deformations can allude to an imminent residual fracture of the material. Therefore, knowledge of the threshold between elastic and plastic deformations is very important. According to Section 3 in [6] this threshold is described by so-called yield criteria. For multi-axile stresses scalar comparison stresses σ_V are introduced. The comparison stress σ_V depends on the stress tensor σ_{ij} and points to beginning yielding if σ_V reaches a critical value σ_{crit} . So yield criteria can

be described by equations of the form $y(\sigma_{ij}) = 0$, where $y(\sigma_{ij}) = \sigma_V(\sigma_{ij}) - \sigma_{\text{crit}}$. For negative values of y the material has solely elastic behavior.

Plastic deformations within grains of a metallic material are caused by dislocation motions where crystal faces are displaced in different directions. Experiments have shown that only shear stresses can plastically deform metallic material. Thus, the deviation of the state of stress from the hydrostatic state of stress $\sigma' = \sigma - \frac{1}{3}\text{tr}(\sigma)$ decides if metal begins to yield.

In the space of principle stresses the yield surface circumscribes a cylinder around the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$. In this work we use the von Mises yield criterion which is given by a cylinder around the hydrostatic axis with a cross section of radius $R = \sqrt{2} k_F$. Here, k_F is the critical value of the von Mises yield criterion

$$\sqrt{\frac{1}{2}\text{tr}(\sigma'^2)} = \sqrt{\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]} = k_F. \quad (2)$$

The left-hand side is proportional to the elastic strain energy of distortion. If the criterion is applied to uniaxial tensile tests the relationship $k_F = R_p/\sqrt{3}$ is obtained, where R_p is the critical value of the only nonzero principle stress in a uniaxial tensile test. Defining the von Mises stress as $\sigma_v = \sqrt{\frac{3}{2}\text{tr}(\sigma'^2)}$ the criterion can be rewritten in the form $\sigma_v = R_p$ and used to predict yielding of metal under any loading condition from results of uniaxial tensile tests.

Now, we present the Ramberg-Osgood equation which is used to locally derive strain levels from scalar comparison stresses which determine strain-controlled fatigue life, confer [31]. This equation describes stress-strain curves of metals near their yield points and is very accurate in the case of smooth elastic-plastic transitions. The latter can be observed for metals that harden with plastic deformations, for example. Let K denote the strain hardening coefficient and n the strain hardening exponent. Then, the Ramberg-Osgood equation is given by

$$\varepsilon_v = \frac{\sigma_v}{E} + \left(\frac{\sigma_v}{K}\right)^{1/n} \quad (3)$$

with Young' modulus $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and the equation defines the comparison strain ε_v . We also write $\varepsilon_v = \text{RO}(\sigma_v)$.

At the end of this section we introduce the method of stress shakedown by Neuber which describes a conversion of elastic to elastic-plastic stress, confer [28] and [6]. If linear elasticity leads to stress values which are greater than the material yield strength such a stress shakedown can be performed to compute elastic-plastic stress values. The stress shakedown is based on an energy-conservation ansatz which results in a relationship between the elastic von Mises stress³ σ_v^e and the elastic-plastic von Mises stress σ_v :

$$\frac{(K_t \sigma_v^e)^2}{E} = \sigma_v \varepsilon_v = \frac{\sigma_v^2}{E} + \sigma_v \left(\frac{\sigma_v}{K}\right)^{1/n}. \quad (4)$$

Here, the Ramberg-Osgood approach is used. K_t is the notch factor, which is set to one if σ_v^e is obtained from the solution of a linear elastic equation (1) where notches are part of the boundary definition. Given the elastic comparison stress σ_v^e , we can thus calculate the elastic-plastic von Mises stress by solving (4) and thus we are able to obtain ε_v from (3). We also write $\sigma_v = \text{SD}^{-1}(\sigma_v^e)$.

³Confer [24] for details on Neuber shakedown in conjunction with equivalent stresses. As an alternative to Neuber's rule we could have also used Glinka's method, confer e.g. [26].

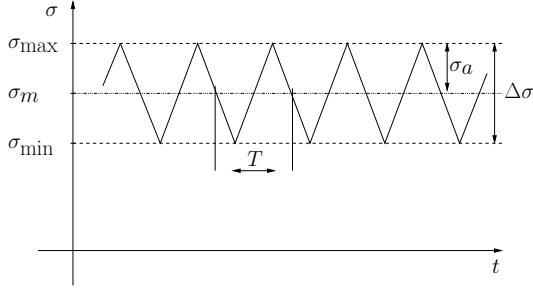


Figure 1: Triangle shaped load-time-curve.

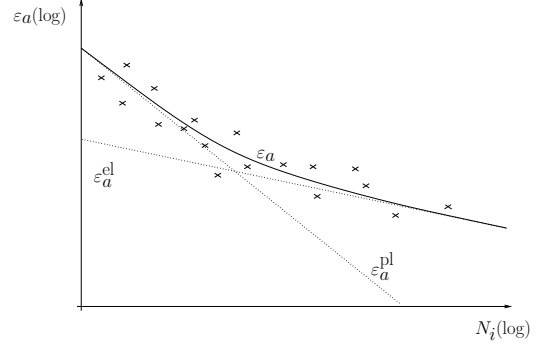


Figure 2: EN-diagram of a standardized specimen.

3 Fatigue

In structural analysis, fatigue describes the damage or failure of material under cyclic loading, confer [6] and [30]. Compared to the static case material is damaged by much lower load amplitudes of cyclic loading. Moreover, in ductile materials under cyclic loading there are less significant plastic deformations before failure so that an imminent damage is more difficult to detect. In many engineering cases the functional description of the loading cycles is very complex. This leads to a preference on constrained analyzes of representative cases of cyclic loading. Figure 1 shows a triangle shaped uniaxial load-time-curve as an example, where $\sigma_a = (\sigma_{\max} - \sigma_{\min})/2$ is the stress amplitude. In material science standardized specimens under such uniaxial load-time-curves are analyzed to predict the behavior of more complex engineering parts.

For backgrounds on surface driven LCF failure mechanism with respect to polycrystalline metal we refer to [6], [30], [15] and [36]. Furthermore, recall our introduction. We now discuss models to describe failure times in fatigue. In fatigue specimen testing the number of cycles until failure is determined and if the tests are strain controlled so-called $E - N$ diagrams are created, see the test points in Figure 2. The functional relationship of the corresponding strain amplitude and number of cycles is called Wöhler curve. For the purpose of application and analysis of fatigue one subdivides the range of cycles in low-cycle fatigue (LCF) and high-cycle fatigue (HCF) although there is no sharp threshold between them, confer [33]. Note that LCF loads are often strain controlled. However, HCF loads are mainly stress controlled so that corresponding $S - N$ diagrams are analyzed.

Figure 2 shows the relationship of strain amplitude ε_a and the life time N_i to crack initiation measured in cycles. In the LCF range the plastic part $\varepsilon_a^{\text{pl}}$ has the greater contribution to the strain amplitude whereas in the HCF range the elastic part $\varepsilon_a^{\text{el}}$ dominates. It is assumed that the additive law $\varepsilon_a = \varepsilon_a^{\text{el}} + \varepsilon_a^{\text{pl}}$ holds. The elastic part can be described with the Basquin equation $\varepsilon_a^{\text{el}} = \frac{\sigma'_f}{E} (2N_i)^b$, where σ'_f is the fatigue strength coefficient, b is the fatigue strength exponent and E is Young's modulus. For the LCF range the Coffin-Manson equation $\varepsilon_a^{\text{pl}} = \varepsilon'_f (2N_i)^c$ describes the denominating plastic part, where ε'_f and c are empirical constants called fatigue ductility coefficient and exponent, respectively. A discussion of the physical origin of this

equation can be found in [36]. One finally obtains the Coffin-Manson-Basquin (CMB) equation

$$\varepsilon_a = \varepsilon_a^{\text{el}} + \varepsilon_a^{\text{pl}} = \frac{\sigma'_f}{E}(2N_i)^b + \varepsilon'_f(2N_i)^c. \quad (5)$$

This equation can be calibrated according to the test points by means of maximum likelihood methods, for example. For this and other methods confer [17] and [33].

Having analyzed standardized specimens one assigns that Wöhler curve to an engineering part under cyclic loading which corresponds to the loading and temperature conditions at the part's surface position of highest stress. Safety factors are additionally imposed to account for the stochastic nature of LCF, size effects⁴ and for uncertainties in the stress and temperature fields. In the following we refer to this method as the standard method in fatigue which is widely used in engineering, confer [30], [6] and [33].

Note that several extensions exist to (5). In particular, when approaching the HCF region at $\sim 10^4 - 10^5$ cycles, mean stress effects are of increasing importance. The modified Morrow equation is one approach that would fit into the approach taken here without changing much of the mathematics. We refer to [30] for further discussions on that topic.

4 From Reliability Statistics to Probabilistic LCF

Since fatigue tests show a great scatter in the life time of specimens several fatigue tests have to be conducted to obtain statistical statements for life times. An overview over fatigue tests and statistical methods can be found in [33] and [30]. In the following we employ the concept of reliability to derive our probabilistic LCF model. From a materials engineering and mathematical point of view our model has significant advantages compared to the standard method in fatigue, recall Section 3. On the one hand it bypasses the standard specimen approach and considers size effects, i.e. results from arbitrary geometries under LCF failure mechanism can be used to calibrate our model and every position of the surface of an engineering part is taken into account by a surface integral which does not need information on Wöhler curves of a specific specimen. Thereby inhomogeneous stress fields are considered. On the other hand our model fits into the setting of sensitivity analysis in shape optimization what we will address in [19].

We model failure-time processes on continuous scale, although time in our context is a number of load cycles and strictly speaking is an integer number. Let N denote a continuous random variable on some probability space with values in \mathbb{R}_0^+ which describes the time of crack initiation here identified with failure of a system or component. Let P be the underlying probability measure and recall the cumulative distribution function $F_N(n) = P(N \leq n)$, the density function $f_N(n) = dF_N(n)/dn$, the hazard function $h(n) = f_N(n)/(1 - F_N(n))$, the cumulative hazard function $H(n) = \int_0^n h(s)ds$ and the survival function $S_N(n) = 1 - F_N(n) = \exp(-H(n))$, confer [13].

Let an LCF failure mechanism on a boundary of a domain $\Omega \subset \mathbb{R}^3$ be given whose failure-time is described by the random variable N . Moreover, let $\{A_i\}_{i=1,\dots,m}$ be an arbitrary partition of $\partial\Omega$, i.e. $\partial\Omega = \cup_{i=1}^m A_i$ with A_i pairwise disjoint. Now, we introduce and discuss the crucial assumption for our local and probabilistic model of LCF in the case of polycrystalline metal:

⁴Note that different geometries of test specimens lead to different Wöhler curves, confer [30].

Assumption (H)

The LCF failure mechanism on $\partial\Omega$ induces failure mechanisms on each A_i with crack initiation times N_i for $i = 1, \dots, m$ such that the random variables $\{N_i\}_{i=1, \dots, m}$ are independent.

The time of initiation of the first LCF surface crack on $\partial\Omega$ is always $N = \min(N_1, \dots, N_m)$. This is identified as the cycles to failure for the component. If h_i denotes the hazard function of N_i for $i = 1, \dots, m$ and assumption (H) holds it follows that $h = \sum_{i=1}^m h_i$ is the hazard function of N . That is why we call the previous assumption the property of spatial additivity of hazard rates. For polycrystalline metal assumption (H) will hold to a good approximation if the surface zone that is affected from the crack initiation process of a single LCF crack is small with respect to the surface of the component. This surface zone corresponds to faces of a few grains. As long range order phenomena are unusual in polycrystalline metal, we pass to the following stronger assumption:

Assumption (L)

In any measurable surface region $A \subseteq \partial\Omega$ the corresponding hazard rate h_A is a local functional of the elastic displacement field u in that particular region,

$$h_A(n) = \int_A \rho(n; \nabla u, \nabla^2 u) dA. \quad (6)$$

Here, ∇u is the Jacobian matrix and $\nabla^2 u$ the Hessian of u . Note that the loads that we consider in the given context (mainly elastic plastic stresses and strains) can all be expressed as functions of ∇u .

Let us motivate that assumption (L) is only slightly stronger than assumption (H). Consider a surface portion $A \subseteq \partial\Omega$ over which the variation of loads is negligible. Then we can further subdivide this region into m smaller pieces A_j of equal surface volume $|A_j| = |A|/m$. As the loads are approximately constant over the A_j , we should have $h_{A_j} \approx h_{A_1}$, hence the hazard functions for crack initiation in the respective portion of the surface are essentially equal. Since for whatever subdivision $h_A = \sum_{j=1}^m h_{A_j}$ by assumption (H), we get that $h_A \approx m \times h_{A_1} = \frac{|A|}{|A_1|} h_{A_1}$. For $m \rightarrow \infty$ we thus see that the limit $\rho = \lim_{m \rightarrow \infty} h_{A_1}/|A_1|$ exists and one obtains $h \approx A \cdot \rho$. Here, ρ stands for a hazard density function.

For mathematical simplicity, we restrict ourselves to most simplistic form of assumption (L), namely the dependence on elastic strains (or equivalently stresses), only. Modeling damage times in the spirit of elastic support factors depending on $\nabla^2 u$ via $\chi^* = \frac{|\nabla \sigma_v|}{\sigma_v}$ might be an interesting option for the future.

In the case of inhomogeneous strain fields assumption (L) implies the previous ansatz to the form $h(n) = \int_{\partial\Omega} \rho(n; \varepsilon^e) dA$ for some hazard density function ρ and where ε^e is the linearized strain rate tensor. Finally, we obtain the cumulative distribution function which yields the probability of failure in $\partial\Omega$ until cycle n :

$$\begin{aligned} F_N(n) &= 1 - \exp(-H(n)) = 1 - \exp\left(-\int_0^n h(t) dt\right) \\ &= 1 - \exp\left(-\int_0^n \int_{\partial\Omega} \rho(t; \varepsilon^e) dA dt\right). \end{aligned} \quad (7)$$

The ansatz (7) can also be derived by the Poisson point process on $[0, \infty] \times \partial\Omega$ with $\rho(n; \varepsilon^e(x))$ as the intensity measure. The advantage of this point of view is that also the probability of

the occurrence of a given number of cracks initiated in $A \subseteq \partial\Omega$ within n load cycles can be calculated via the Poisson statistics as

$$P(\text{number of crack initiations on } A = q) = e^{-z} \frac{z^q}{q!} \text{ for } z = \int_0^n \int_A \rho(t; \varepsilon^e) dAdt. \quad (8)$$

But this approach will break down, if cracks have grown sufficiently large in order to mutually influence their local stress fields. For an introduction into point processes confer [5] and [25].

Not having specified the hazard density function ρ we now establish a link to deterministic LCF analysis which leads to an appropriate choice for ρ . Recall the CMB equation (5) which determines the functional dependency of strain and cycle of failure for a standard specimen of a fatigue test, confer Section 3 and [30]. The relation (5) refers to a specimen with a specific geometry in a homogeneous strain field. In order to convey our approach so far to an arbitrary geometry in inhomogeneous strain fields one has to interpret the CMB ansatz in a new way, but such that the results for standard specimens under homogeneous conditions are included.

In order to be concrete, we will make the assumption that life cycles to crack initiation N are Weibull distributed:

$$\rho(n; x) = \rho(n; \varepsilon^e(x)) = \frac{m}{N_{\det}(\varepsilon^e(x))} \left(\frac{n}{N_{\det}(\varepsilon^e(x))} \right)^{m-1}. \quad (9)$$

Here, $N_{\det}(\varepsilon^e)$ is the scale parameter and is computed as follows: Solving the elasticity problem (1) results in the linearized strain rate tensor ε^e and the linear elastic von Mises stress field σ_v^e . Applying the method of stress shakedown by Neuber to $\sigma_v^e/2$ yields the elastic-plastic von Mises stress amplitude σ_a corresponding to uniaxial fatigue test. Using the Ramberg-Osgood equation yields the corresponding strain field ε_a which is the left-hand side of the CMB equation (5). The latter equation finally leads to $N_{\det}(\varepsilon^e)$. The shape parameter $m \in (0, \infty)$ controls the scatter with small $m > 1$ corresponds⁵ to large scatter and the limit $m \rightarrow \infty$ is the deterministic limit. The Weibull model is not necessarily true for real data. However, the Weibull hazard rate can be easily replaced by any other differentiable hazard rate with scale parameter N_{\det} , without changing much of the content of this paper. Combining (7) and (9) yields

$$F_N(n) = 1 - \exp \left(- \int_0^n \int_{\partial\Omega} \frac{m}{N_{\det}} \left(\frac{s}{N_{\det}} \right)^{m-1} dAds \right) = 1 - \exp \left(- \left\| \frac{n}{N_{\det}} \right\|_{L^m(\partial\Omega, \mathbb{R})}^m \right). \quad (10)$$

Definition 4.1 (Local and Probabilistic Model for LCF)

Let $\Omega \subset \mathbb{R}^3$ be a domain whose boundary $\partial\Omega$ is subject to surface driven LCF failure mechanism. Moreover, let the scale field $N_{\det}(x) = N_{\det}(\varepsilon_a(x))$, $x \in \partial\Omega$, be the solution of the CMB equation (5)

$$\varepsilon_a(x) = \frac{\sigma_f'}{E} (2N_{\det}(x))^b + \varepsilon_f' (2N_{\det}(x))^c, \quad (11)$$

where $\varepsilon_a(x)$ is calculated from $\varepsilon^e(x)$ via⁶ linear isotropic elasticity, from the von Mises stress $\sigma_v(x)$, from $\sigma_a(x) = \text{SD}^{-1}(\sigma_v(x)/2)$ according to the method of stress shakedown by Neuber and from the Ramberg-Osgood equation with $\varepsilon_a(x) = \text{RO}(\sigma_a(x))$. Then, the local and probabilistic model for LCF is defined by the cumulative distribution function (10) for $n \in \mathbb{R}_0^+$ and some $m \geq 1$, which yields the probability for LCF crack initiation in the interval $[0, n]$.

⁵ $0 < m \leq 1$ is not realistic for fatigue.

⁶Confer Sections 2 and 3.

Note that CMB parameters in this approach are not the same as obtained from fitting standard specimen data. In the local, probabilistic approach to LCF the CMB parameters need to be fixed by the usual maximum likelihood methods of reliability statistics, confer [17] and [13]. Volume driven HCF failure mechanism can be considered by replacing the surface integral in (10) with a volume integral whose integrand only differs by different material parameters. For a discussion of HCF failure mechanism confer [6] and [30]. In case of mixed LCF and HCF failure mechanism a volume integral of similar form can added to (10).

In contrast to the probabilistic model the standard method in fatigue yields the following deterministic LCF model:

Definition 4.2 (*Deterministic Model for LCF*)

Given the setting of the previous Definition 4.1 the deterministic model for LCF is determined by $\inf\{N_{\text{det}}(x)|x \in \partial\Omega\}$.

In the next sections we will prove the existence of optimal shapes for Definitions 4.1 and 4.2.

5 Shape Optimization and C^k -Admissible Domains

In the following we introduce into an abstract setting of shape optimization and discuss so-called C^k -admissible domains. This leads to a theoretical frame for an existence proof of optimal designs with respect to a general class of cost-functionals which include functionals of LCF and HCF failure mechanism in our probabilistic and deterministic sense. First, we address basic notations and closely follow Section 2.4 in [20]. For further introductions into shape optimization confer [35], [8] and [32], for example.

Notation (Family of Admissible Domains, State Space)

Let $\tilde{\mathcal{O}}$ denote a family of admissible domains and let $V(\Omega)$ for every $\Omega \in \tilde{\mathcal{O}}$ denote a state space of real functions defined in Ω .

Notation (Convergence of Sets and of Functions with Variable Domains)

Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{O}}$ and let $\Omega \in \tilde{\mathcal{O}}$. Then $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$ as $n \rightarrow \infty$ denotes the convergence of $(\Omega_n)_{n \in \mathbb{N}}$ against Ω . If $(y_n)_{n \in \mathbb{N}}$ is a sequence of functions with $y_n \in V(\Omega_n)$ for every $n \in \mathbb{N}$ and if $y \in V(\Omega)$ then $y_n \rightsquigarrow y$ as $n \rightarrow \infty$ denotes the convergence of $(y_n)_{n \in \mathbb{N}}$ against y . Moreover, it is assumed that any subsequence of a convergent sequence converges against the limit of the original one.

In every $\Omega \in \tilde{\mathcal{O}}$ one solves a state problem which can be a PDE or a variational inequality, for example. Assuming that every state problem has a unique solution and associating with any $\Omega \in \tilde{\mathcal{O}}$ the corresponding unique solution $u(\Omega) \in V(\Omega)$ one obtains the map $u : \Omega \mapsto u(\Omega) \in V(\Omega)$. Let \mathcal{O} be a subfamily of $\tilde{\mathcal{O}}$, then $\mathcal{G} = \{(\Omega, u(\Omega)) | \Omega \in \mathcal{O}\}$ is called the graph of the mapping $(u(\cdot))$ restricted to \mathcal{O} .

Definition 5.1 (*Cost Functional, Optimal Shape Design Problem*)

A cost functional J on $\tilde{\mathcal{O}}$ is given by a map $J : (\Omega, y) \mapsto J(\Omega, y) \in \mathbb{R}$, where $\Omega \in \tilde{\mathcal{O}}$ and $y \in V(\Omega)$. Let \mathcal{O} be a subfamily of $\tilde{\mathcal{O}}$ and for every $\Omega \in \mathcal{O}$ let $u(\Omega)$ be the unique solution of a state problem given in Ω . An optimal shape design problem can then be defined by

$$\begin{cases} \text{Find } \Omega^* \in \mathcal{O} \text{ such that} \\ J(\Omega^*, u(\Omega^*)) \leq J(\Omega, u(\Omega)) \quad \forall \Omega \in \mathcal{O}. \end{cases} \quad (12)$$

Now, we present a statement regarding the existence of optimal shapes. Note that this theorem will structure the following section where we prove of our main existence results.

Theorem 5.2 (*Existence of An Optimum in Shape Design Problems*)

Let $\tilde{\mathcal{O}}$ be a family of admissible domains and \mathcal{O} a subfamily. Moreover, let J be a cost functional on $\tilde{\mathcal{O}}$ and assume that every $\Omega \in \tilde{\mathcal{O}}$ has a state problem with state space $V(\Omega)$ where each such state problem has a unique solution $u(\Omega) \in V(\Omega)$. Finally, conjecture

- Compactness of $\mathcal{G} = \{(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$:

Every sequence $(\Omega_n, u(\Omega_n))_{n \in \mathbb{N}} \subset \mathcal{G}$ has a subsequence $(\Omega_{n_k}, u(\Omega_{n_k}))_{n_k \in \mathbb{N}}$ which satisfies

$$\begin{aligned} \Omega_{n_k} &\xrightarrow{\tilde{\mathcal{O}}} \Omega, \quad k \rightarrow \infty, \\ u(\Omega_{n_k}) &\rightsquigarrow u(\Omega), \quad k \rightarrow \infty \end{aligned}$$

for some $(\Omega, u(\Omega)) \in \mathcal{G}$.

- Lower semi-continuity of J :

Let $(\Omega_n)_{n \in \mathbb{N}}$ with $\Omega_n \in \tilde{\mathcal{O}}$, $n \in \mathbb{N}$, and $(y_n)_{n \in \mathbb{N}}$ with $y_n \in V(\Omega_n)$, $n \in \mathbb{N}$, be sequences and let Ω and y be some elements in $\tilde{\mathcal{O}}$ and in $V(\Omega)$, respectively. Then

$$\left. \begin{aligned} \Omega_n &\xrightarrow{\tilde{\mathcal{O}}} \Omega, \quad n \rightarrow \infty, \\ y_n &\rightsquigarrow y, \quad n \rightarrow \infty \end{aligned} \right\} \implies \liminf_{n \rightarrow \infty} J(\Omega_n, y_n) \geq J(\Omega, y).$$

Under these assumptions the optimal shape design problem (12) possesses at least one solution.

According to the previous theorem and abstract setting we have to define the family of admissible domains $\tilde{\mathcal{O}}$. In this work we consider so-called C^k -admissible domains which have smooth boundaries. For the sake of simplicity we only optimize a part of the boundary of these shapes. This method is described in Section 2.8 of [35] and in [20], too. Importantly, C^k -admissible domains satisfy compactness properties as required in Theorem 5.2 according to the Arzela-Ascoli theorem.

At first, admissible domains are defined which are determined by uniformly bounded functions. Later on, these functions are assumed to be sufficiently smooth to obtain C^k -admissible domains. On these domains we will later impose boundary value problems of linear elasticity which are characterized by disjoint Dirichlet and Neumann boundaries. In the following we employ so-called uniform cone properties which are defined in Appendix A of [20].

Definition 5.3 (*Basic Design, Design Variables, Admissible Domains*)

Let $\hat{\Omega} \subset \mathbb{R}^3$ be a simply connected and bounded domain which is called basic design under the following assumptions:

- $\hat{\Omega}$ has a uniform cone property and a C^k -boundary⁷ for some $k \in \mathbb{N}, k \geq 1$.
- There is some $\alpha_{\min} \in \mathbb{R}$ so that the cross section $\Omega_{2d} = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, \alpha_{\min}) \in \hat{\Omega}\}$ is a nonempty domain in \mathbb{R}^2 .

⁷This leads to a local description of $\partial\hat{\Omega}$ by a finite number of hemisphere transformations of class C^k , confer Definitions A.1 and A.2 and Remark A.3.

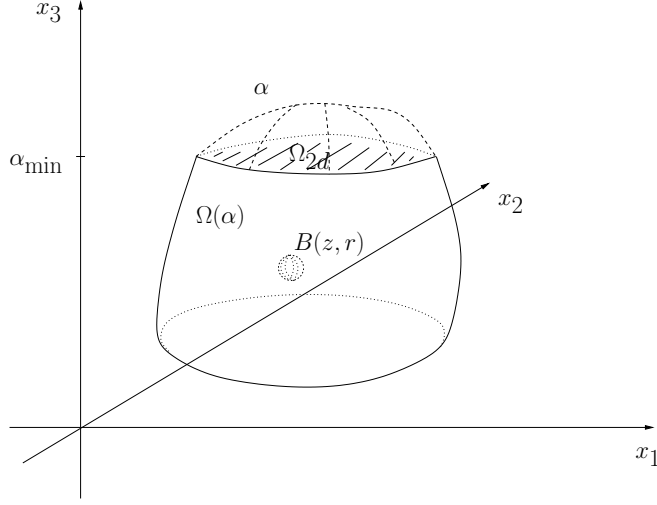


Figure 3: Admissible shape.

- There are some $z \in \hat{\Omega}$ and $r > 0$ such that $B(z, r) \subset \hat{\Omega}$ and $z_3 + r < \alpha_{\min}$.

For $\alpha_{\max} > \alpha_{\min}$ and positive constants L_1, L_2, L_3 the elements of

$$\tilde{U}^{\text{ad}} = \left\{ \alpha \in C^k(\overline{\Omega}_{2d}) \mid \alpha_{\min} \leq \alpha \leq \alpha_{\max} \text{ in } \overline{\Omega}_{2d}, \alpha|_{\partial\Omega_{2d}} = \alpha_{\min}, \int_{\Omega_{2d}} \alpha(x) dx = L_1, \|\alpha\|_{C^k} \leq L_2, \right. \\ \left. \left| \alpha^{(k)}(x) - \alpha^{(k)}(y) \right| \leq L_3 \|x - y\|_2 \quad \forall x, y \in \overline{\Omega}_{2d} \right\}$$

are called design variables. Let $\alpha \in \tilde{U}^{\text{ad}}$ define the set

$$\Omega(\alpha) = \{x \in \hat{\Omega} \mid x_3 \leq \alpha_{\min}\} \setminus \overline{B(z, r)} \cup \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \Omega_{2d}, \alpha_{\min} < x_3 < \alpha(x_1, x_2)\},$$

see Figure 3. Varying only functions $\alpha \in \tilde{U}^{\text{ad}}$ the domains $\tilde{\mathcal{O}} = \{\Omega(\alpha) \mid \alpha \in \tilde{U}^{\text{ad}}\}$ with Lipschitz continuous boundaries are called admissible domains. Finally, choose a superset Ω^{ext} such that extension Lemma A.4 can be applied.

Lemma 5.4 \tilde{U}^{ad} is compact in $(C^k(\overline{\Omega}_{2d}), \|\cdot\|_{C^k})$.

Proof: Applying the Arzela-Ascoli theorem several times to

$$\left\{ \alpha \in C^k(\overline{\Omega}_{2d}) \mid \|\alpha\|_{C^k} \leq L_2, \left| \alpha^{(k)}(x) - \alpha^{(k)}(y) \right| \leq L_3 \|x - y\|_2 \quad \forall x, y \in \overline{\Omega}_{2d} \right\} \quad (13)$$

shows the compactness in $(C^k(\overline{\Omega}_{2d}), \|\cdot\|_{C^k})$. \tilde{U}^{ad} being a closed subset of (13) proves the statement of this lemma. \blacksquare

Definition 5.5 (C^k -Convergence of Sets)

$\Omega(\alpha_n) \xrightarrow{\tilde{\mathcal{O}}} \Omega(\alpha)$ as $n \rightarrow \infty$ is defined by $\alpha_n \rightarrow \alpha$ in $C^k(\overline{\Omega}_{2d})$ as $n \rightarrow \infty$, where α, α_n and $\Omega(\alpha), \Omega(\alpha_n)$ for $n \in \mathbb{N}$ are defined as in Definition 5.3.

In order to apply regularity results of linear elasticity we have to require a sufficiently smooth boundary of the admissible domains. Therefore, additional boundary conditions on the design variables α are introduced which enable the construction of such domains.

Definition 5.6 (*C^k -Admissible Domains*)

Let \tilde{U}^{ad} be the set of design variables of Definition 5.3 and let $S_\beta : \partial\Omega_{2d} \rightarrow \mathbb{R}$ be functions for multi-indices β with $1 \leq |\beta| \leq k$. Define $U^{\text{ad}} = \left\{ \alpha \in \tilde{U}^{\text{ad}} \mid \nabla^\beta \alpha|_{\partial\Omega_{2d}} = S_\beta \ \forall |\beta| \in \{1, \dots, k\} \right\}$. Choosing S_β so that $\Omega(\alpha)$ has a C^k -boundary for every $\alpha \in U^{\text{ad}}$ the set $\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in U^{\text{ad}}\}$ denotes the family of so-called C^k -admissible domains.

Lemma 5.7 *U^{ad} is compact in $(C^k(\overline{\Omega}_{2d}), \|\cdot\|_{C^k})$.*

Proof: Note that U^{ad} is a closed subset of \tilde{U}^{ad} , where \tilde{U}^{ad} is already compact according to Lemma 5.4. ■

6 Existence of Optimal Shapes

We consider optimal shape design problems in linear elasticity which in particular include shape optimization problems given by the cost functional (10) of our local and probabilistic model for LCF. The state problems are described by mixed problems of linear elasticity. We analyze a very general class of cost functionals which are not constraint by convexity assumptions and we optimize shapes within the family of C^k -admissible domains. As already announced in the previous section only a part of the boundary is subject of optimization. The abstract setting of Section 5 and Theorem 5.2 determine the structure of this section which leads to our existence results. Therefore, the technical steps will be addressed here in the following order:

- Linear elasticity and regularity properties.
- State equation and space given by the mixed problem of linear elasticity.
- Convergence of functions with variable C^k -admissible domains.
- Compactness shown by the Arzela-Ascoli theorem and by Schauder estimates.
- Class of cost functionals and continuity properties.
- Proof of the main existence result.
- Applying the existence result to our probabilistic and deterministic models of fatigue.

At first, we outline important results in linear elasticity which will influence our choice of state space and of the definition of convergence of functions with variable domains. Recall the mixed problem (1) which is called pure displacement problem if $\partial\Omega = \partial\Omega_D$ and pure traction problem if $\partial\Omega = \partial\Omega_N$. But in the following we consider the so-called disjoint displacement-traction problem where $\partial\Omega_D$ and $\partial\Omega_N$ are disjoint.

Since we assume a homogeneous temperature field the Lamé coefficients λ and μ are constant. Regularity results for the mixed problem depend crucially on the properties of the domain's boundary in which the elasticity equations are posed. The following statement of [10] ensures the existence of a weak solution of the mixed problem.

Theorem 6.1 (*Existence of a Weak Solution*)

Let $\Omega \subset \mathbb{R}^n$ be a domain and $\partial\Omega_D \subset \partial\Omega$ be measurable where $\partial\Omega_D$ has a positive area. Let

the Lamé coefficients λ, μ be positive constants and let $f \in [L^{6/5}(\Omega)]^3$, $g \in [L^{4/3}(\partial\Omega_N)]^3$ where $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. Moreover, define on $V_{DN} = \{v \in [H^1(\Omega)]^3 \mid v = 0 \text{ a.e. on } \partial\Omega_D\}$:

$$B(u, v) = \int_{\Omega} \lambda \nabla u \cdot \nabla v \, dx + \int_{\Omega} 2\mu \left(\sum_{i,j=1}^3 \varepsilon_{ij}(u) \varepsilon_{ij}(v) \right) dx,$$

$$L(v) = \int_{\Omega} f \cdot v \, dx + \int_{\partial\Omega_N} g \cdot v \, dA.$$

Then, there exists a unique $u \in V_{DN}$ that satisfies

$$B(u, v) = L(v) \quad \forall v \in V_{DN} \quad (14)$$

and additionally $J(u) = \inf\{J(v) \mid v \in V_{DN}\}$, where $J(v) = \frac{1}{2}B(v, v) - L(v)$.

The unique solution $u \in V_{DN}$ of (14) is called the weak solution of (1). The following inequality can be found in [7]: If $\Omega \subset \mathbb{R}^3$ is a domain the so-called Korn's second inequality

$$c\|v\|_{[H_0^1(\Omega)]^3} \leq \left(\int_{\Omega} \sum_{ij} \varepsilon^e(v)_{ij} \varepsilon^e(v)_{ij} dx \right)^{1/2} = \|\varepsilon^e(v)\|_{H^0(\Omega)} \quad (15)$$

holds for all $v \in V_{DN}$. The disjoint displacement-traction problem has additional regularity properties if $\partial\Omega$ and the forces f and g are sufficiently regular, confer Section 6.3 of [10]:

Theorem 6.2 (Regularity of the Weak Solution for the Disjoint Displacement-Traction Problem)

Let $\Omega \subset \mathbb{R}^3$ be a domain with a C^4 -boundary, let $f \in [W^{2,p}(\Omega)]^3$ and let $g \in [W^{1-1/p,p}(\partial\Omega)]^3$ for some $p \geq 6/5$. Consider on Ω a disjoint displacement-traction problem. Then, there exists a unique solution $u \in V_N$ of $B(u, v) = L(v)$ for all $v \in V_N$, where V_N, B and L are defined as in Theorem 6.1. Moreover, u is an element of $[W^{4,p}(\Omega)]^3$.

The key to the proof is to employ the fact that the previous problems (pure displacement, pure traction, disjoint displacement-traction problem) are uniformly elliptic and satisfy the so-called supplementary and complementing conditions. These conditions are introduced in [2] where Schauder estimates are also described in detail which can be applied to solutions of mixed problems. The Schauder estimates are an important ingredient in our proof of existence for optimal shapes in this section.

In the following we consider on C^4 -admissible shapes⁸ $\Omega(\alpha) \in \mathcal{O}$ the state problem which is given by the mixed problem, here in a rewritten form of (1):

$$\lambda \sum_{j=1}^3 \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \sum_{j=1}^3 \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right) = -f_i \quad \text{in } \Omega(\alpha), \quad i = 1, 2, 3,$$

$$u_i = 0 \quad \text{on } \partial\Omega(\alpha)_D, \quad i = 1, 2, 3, \quad \mathcal{P}(\alpha)$$

$$\lambda \nu_i \sum_{j=1}^3 \frac{\partial u_j}{\partial x_j} + \mu \sum_{j=1}^3 \nu_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = g_i \quad \text{on } \partial\Omega(\alpha)_N, \quad i = 1, 2, 3,$$

⁸ C^4 is needed for Theorem 6.2. Therefore, set $k = 4$ in the definition of \mathcal{O} .

where $\partial\Omega(\alpha)_D = \partial B(z, r)$ is the complete interior boundary of $\Omega(\alpha)$, $\partial\Omega(\alpha)_N = \partial\Omega(\alpha) \setminus \partial\Omega(\alpha)_D$ the exterior boundary⁹ and where ν is the normal of $\partial\Omega_N(\alpha)$. See Figure 3, too. We choose $V(\Omega(\alpha)) = [C^3(\overline{\Omega(\alpha)})]^3$ as state space for $\mathcal{P}(\alpha)$ so that we can additionally analyze stress gradients in our fatigue models. Therefore, we employ the following definition with $q = 3$ for the convergence of functions with variable domains in \mathcal{O} , also confer Section 2.5.2 in [20].

Definition 6.3 (*C^q -Convergence of Functions with Variable Domains*)

Recalling the sets $\tilde{\mathcal{O}}$ and Ω^{ext} of Definition 5.3 let $p_\Omega : [C^q(\overline{\Omega})]^3 \rightarrow [C_0^q(\Omega^{\text{ext}})]^3$ be the extension operator which can be derived from Lemma A.4 for $q \in \mathbb{N} \setminus \{0\}$. For $u \in [C^q(\overline{\Omega})]^3$ set $u^{\text{ext}} = p_\Omega u$. For $(\Omega_l)_{l \in \mathbb{N}} \subset \tilde{\mathcal{O}}$, $\Omega \in \tilde{\mathcal{O}}$ and $(u_l)_{l \in \mathbb{N}}$ with $u_l \in [C^q(\overline{\Omega_l})]^3$, $l \in \mathbb{N}$, and $u \in [C^q(\overline{\Omega})]^3$ the expression $u_l \rightsquigarrow u$ as $l \rightarrow \infty$ is defined by $u_l^{\text{ext}} \rightarrow u^{\text{ext}}$ in $[C_0^q(\Omega^{\text{ext}})]^3$.

We want to find an optimal shape $\Omega(\alpha) \in \mathcal{O}$ which minimizes a functional of the form $J(\Omega, u) = J_{\text{vol}}(\Omega, u) + J_{\text{sur}}(\Omega, u)$ with

$$J_{\text{vol}}(\Omega, u) = \int_{\Omega} \mathcal{F}_{\text{vol}}(x, u(x), \nabla u(x), \nabla^2 u(x)) dx, \quad J_{\text{sur}}(\Omega, u) = \int_{\partial\Omega} \mathcal{F}_{\text{sur}}(\cdot, u(\cdot), \nabla u(\cdot), \nabla^2 u(\cdot)) dA$$

and u uniquely given by $\Omega(\alpha)$ as the solution of the state problem $\mathcal{P}(\alpha)$. The uniqueness is realized by introducing appropriate assumptions with respect to the setting of $\mathcal{P}(\alpha)$ what is shown in the following. As an important result we use the Schauder estimates of Theorem 9.3 in [2] and validate if the corresponding assumptions are satisfied. At first, we present two lemmas which show the existence of sufficiently regular hemisphere transformations – confer Definition A.1 – and the validity of a certain inequality, respectively. These two lemmas will be used in the proof for the next theorem. In the following statements Banach spaces $C^{q,\phi}$ for $\phi \in (0, 1)$ occur whose definition can be found in Section 1 of [3] and in Section 7 of [2].

Lemma 6.4 *Each $\Omega \in \mathcal{O}$ satisfies a hemisphere property where the corresponding hemisphere transformations are of class $C^{3,\phi}$ for $\phi \in (0, 1)$ and have a uniform bound κ with respect to \mathcal{O} .*

Proof: Regarding Definition A.1 we have to show that every $x \in \Omega$ within a certain distance $d > 0$ of $\partial\Omega$ has a neighborhood U_x with $B(x, d/2) \subset U_x$ and

$$\overline{U_x} \cap \overline{\Omega} = T_x(\Sigma_{R(x)}), \quad 0 < R(x) \leq 1, \quad \overline{U_x} \cap \partial\Omega = T_x(F_{R(x)}),$$

for some hemisphere¹⁰ $\Sigma_{R(x)}$ and transformations T_x, T_x^{-1} of class $C^{3,\phi}$. At first we consider $x \in \Omega$ within a sufficiently small distance $d > 0$ of $\Gamma(\alpha)$ where $\Gamma(\alpha)$ denotes the portion of $\partial\Omega = \partial\Omega(\alpha)$ which is determined by the design variable α , see Figure 4:

Because of the definition of the basic design and of the admissible shapes $\alpha \in U^{ad}$ there is a C^4 -extension $\alpha^{\text{ext}} : \Omega_{2d}^{\text{ext}} \rightarrow \mathbb{R}$ of α with $\Omega_{2d} \subset \Omega_{2d}^{\text{ext}}$ which describes a portion $\Gamma^{\text{ext}}(\alpha)$ of the boundary $\partial\Omega(\alpha)$ beyond $\Gamma(\alpha)$ and where Ω_{2d}^{ext} is the image of a C^4 -diffeomorphism $\tilde{T}_{2d} : B(0, \tilde{R}) \subset \mathbb{R}^2 \rightarrow \Omega_{2d}^{\text{ext}}$ for some $\tilde{R} > 0$. This extension is needed in order to consider all $x \in \Omega(\alpha)$ within a sufficiently small distance $d > 0$ of $\Gamma(\alpha)$. Now, we are able to define a hemisphere transformation with the required properties:

$$T_x : \Sigma_{\tilde{R}} \rightarrow \mathbb{R}^2, \quad T_x(x_1, x_2, x_3) = \begin{pmatrix} (\tilde{T}_{2d}(x_1, x_2))_1 \\ (\tilde{T}_{2d}(x_1, x_2))_2 \\ \alpha^{\text{ext}}((\tilde{T}_{2d}(x_1, x_2))_1, (\tilde{T}_{2d}(x_1, x_2))_2) - x_3 \end{pmatrix}, \quad (16)$$

⁹Note that this decomposition of the boundary depends continuously on $\alpha \in U^{ad}$ and the two-dimensional Lebesgue measure of $\partial\Omega(\alpha)_D$ is greater than a positive constant for all $\alpha \in U^{ad}$.

¹⁰ $F_{R(x)}$ denotes the flat boundary of the hemisphere $\Sigma_{R(x)}$.

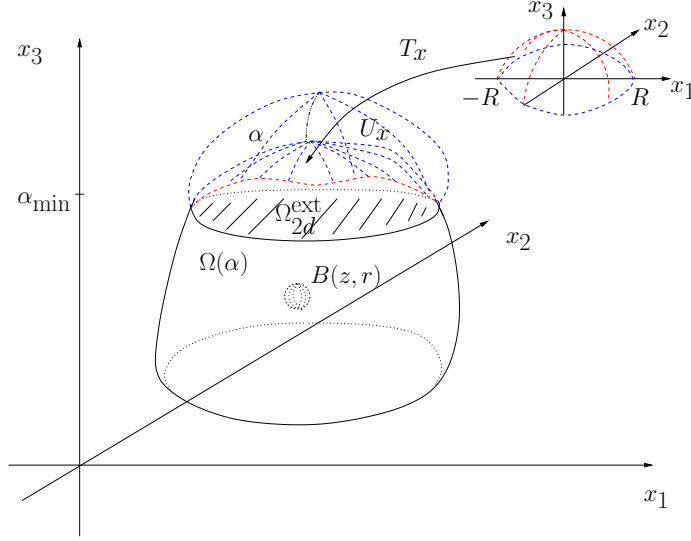


Figure 4: Hemisphere transformation.

see Figure 4, too. The neighborhood U_x can be chosen so that $\overline{U}_x \cap \overline{\Omega} = T_x(\Sigma_{R(x)})$ and $\overline{U}_x \cap \partial\Omega = T_x(F_{R(x)})$. This can be achieved by sufficiently expanding $T_x(\Sigma_{R(x)})$ beyond α^{ext} . Because of the definition of the basic design and of the design variables α one can find a bound for the norms of the hemisphere transformations which is valid for all $\alpha \in U^{ad}$. Analogously the remaining hemisphere transformations can be constructed which are of a finite number and all have a uniform bound denoted by κ . ■

The following lemma contains an inequality which can be found in Section 7 of [1]. Using only a few additional technical arguments a statement about the inequality's constant C can be added regarding its dependency on cone properties of the underlying domain Ω .

Lemma 6.5 *Let \mathcal{M} be a set of bounded domains in \mathbb{R}^n with a uniform cone property and let $\Omega \in \mathcal{M}$. Then, for every $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ uniform with respect to \mathcal{M} such that $\|v\|_{C^0(\Omega)} \leq \varepsilon \|v\|_{C^1(\Omega)} + C \int_{\Omega} |v| dx$ holds for all $v \in C^1(\Omega)$.*

Before applying the Schauder estimates of Theorem 9.3 in [2] to the solutions of $\mathcal{P}(\alpha)$ we introduce the following technical notations: For arbitrary multi-indices ϱ and β define $\delta_{\varrho\beta}$ to be one if $\varrho = \beta$ and to be zero in any other case. A vector and a matrix of multi-indices is given by $\varrho(i)$ and $\beta(ij)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, respectively, where each $\varrho(i)$ and $\beta(ij)$ represents a multi-index. The k -th component is given by $\varrho(i)_k$ and $\beta(ij)_k$, respectively.

Theorem 6.6 *Let the boundary value problem $\mathcal{P}(\alpha)$ be given on a domain $\Omega(\alpha) \in \mathcal{O}$. Suppose that the Lamé coefficients are constants. Moreover, let $f_i \in C^{1,\phi}(\overline{\Omega}^{ext})$ and $g_i \in C^{2,\phi}(\partial\Omega(\alpha))$ for $i = 1, 2, 3$ and $\phi \in (0, 1)$ and let their corresponding norms be bounded by a constant k_{fg} which is independent of α . Then, there are hemisphere transformations of class $C^{3,\phi}$ with uniform bound κ and a unique solution $u \in V(\Omega(\alpha))$ of the boundary value problem $\mathcal{P}(\alpha)$*

which also belongs to $[C^{3,\phi}(\overline{\Omega(\alpha)})]^3$ and satisfies

$$\|u_j\|_{C^{3,\phi}(\Omega)} \leq C \left(\sum_{i=1}^3 \|f_i\|_{C^{1,\phi}(\Omega)} + \sum_{h=1}^3 \|g_h\|_{C^{2,\phi}(\partial\Omega)} + \sum_{k=1}^3 \|u_k\|_{C^0(\Omega)} \right), \quad j = 1, 2, 3, \quad (17)$$

for some constant C . The terms $\|u_k\|_{C^0(\Omega)}$ can be replaced by $\int_{\Omega} |u_k| dx$ for $k = 1, 2, 3$ and C can be chosen uniformly with respect to \mathcal{O} .

Proof: The existence of a unique solution $u \in V(\Omega(\alpha))$ is a consequence of Theorem 6.2: Because $\Omega(\alpha)$ has C^4 -boundary we have a unique solution $u \in [W^{4,p}(\Omega(\alpha))]^3$ for arbitrary $p \geq 6/5$. Then, the general Sobolev inequalities – confer Section 5.6 in [14] – will lead to $u \in [C^{3,\phi}(\overline{\Omega(\alpha)})]^3$ for some $\phi \in (0, 1)$ if p is sufficiently large¹¹.

Now, we show that the assumptions for the Schauder estimates of Theorem 9.3 in [2] are satisfied. Main assumptions are complementing and supplementary boundary conditions, uniform ellipticity with the corresponding constant A , a positive minor constant $\Delta_{\partial\Omega}$ and the existence of hemisphere transformations¹² of class $C^{3,\phi}$ with corresponding norms uniformly bounded by a constant κ , confer Sections 1, 2, 7 and 9 of [2] for a detailed description. We start by rewriting the equations of $\mathcal{P}(\alpha)$ in the form of $\sum_{j=1}^3 l_{ij}(x, \nabla) u_j(x) = f_i(x)$ for $x \in \Omega$ and of $\sum_{j=1}^3 B_{hj}(x, \nabla) u_j(x) = g_h(x)$ for $x \in \partial\Omega_N$, see (1.1) and (2.1) in [2]. Therefore, we write

$$\begin{aligned} l_{ij}(x, \Xi) &= \sum_{|\varrho|=0}^2 a_{ij,\varrho}(x) \Xi^\varrho \\ &= \sum_{|\varrho|=2} \left[\mu(\delta_{ij}(\delta_{\varrho(2,0,0)} + \delta_{\varrho(0,2,0)} + \delta_{\varrho(0,0,2)}) + \delta_{\varrho\gamma(ij)}) + \lambda \sum_{\beta(i)_i > 0} \delta_{\varrho\beta(i)} \right] \Xi^\varrho, \end{aligned} \quad (18)$$

where $\gamma(ij) = (\delta_{1i} + \delta_{1j}, \delta_{2i} + \delta_{2j}, \delta_{3i} + \delta_{3j})$ and where $\sum_{\beta(i)_i > 0}$ is the sum over all multi-indices with $\beta(i)_i > 0$. Corresponding to the Neumann boundary conditions on $\partial\Omega_N$ we write

$$B_{hj}(x, \Xi) = \sum_{|\varrho|=0}^1 b_{hj,\varrho} \Xi^\varrho = \sum_{|\varrho|=1} (\lambda \nu_h(x) \delta_{\varrho\beta(j)} + \delta_{hj} \mu \nu_\varrho(x) + \mu \nu_j(x) \delta_{\varrho\beta(h)}) \Xi^\varrho, \quad (19)$$

where $\beta(1) = (1, 0, 0)$, $\beta(2) = (0, 1, 0)$, $\beta(3) = (0, 0, 1)$, $\nu_{(1,0,0)} = \nu_1$, $\nu_{(0,1,0)} = \nu_2$, $\nu_{(0,0,1)} = \nu_3$. Regarding Theorem 9.3 in [2] and equations (18) and (19) we have $s_i = 0$, $t_j = 2$ and $r_h = -1$ for the Neumann condition and $r_h = -2$ for the Dirichlet condition¹³ which implies $l_0 = \max\{0, r_h\} = 0$ and allows to choose $1 = l \geq l_0$ for $i, j, h \in \{1, 2, 3\}$.

Consider that the complementing conditions and the supplementary boundary conditions are satisfied, confer Section 6.3 of [10]. The existence of appropriate hemisphere transformations of class $C^{3,\phi}$ is shown in Lemma 6.4 where the constants d and κ can be chosen uniformly with respect to \mathcal{O} . According to Section I.3 of [37] the minor constant $\Delta_{\partial\Omega}$ is positive and determined by the Lamé coefficients which are constant in our case.

¹¹Note that f_i and g_i , $i = 1, 2, 3$, are continuously differentiable.

¹²Recall Definition A.1 where also constant d is defined.

¹³Note that homogeneous Dirichlet conditions on $\partial\Omega_D$ lead to $B_{hj}(x, \Xi) = \delta_{hj}$, $g_h(x) = 0$ for all $x \in \partial\Omega_D$ and to $r_h = -2$.

Now, we analyze the effect of the hemisphere transformation T_x on the ellipticity constant and follow Section 9 of [2]. Let $c(\nabla) = \sum_{\phi} c_{\phi} \nabla^{\phi} = \sum_{i_1, i_2, \dots} c_{i_1, i_2, \dots} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \dots$ be an arbitrary linear combination of differentiation operators. It is transformed into $\hat{c}(\hat{\nabla}) = \sum_{i_1, i_2, \dots} \hat{c}_{i_1, i_2, \dots} \frac{\partial^{i_1}}{\partial \hat{x}_1^{i_1}} \frac{\partial^{i_2}}{\partial \hat{x}_2^{i_2}} \dots$, where $\frac{\partial}{\partial \hat{x}_j} = \sum_i \frac{\partial x_i}{\partial \hat{x}_j} \frac{\partial}{\partial x_i}$ and each $\hat{c}_{i_1, i_2, \dots}$ is a linear combination of the $c_{i_1, i_2, \dots}$ with coefficients that are products of the $\frac{\partial \hat{x}_j}{\partial x_i}$. Correspondingly, we obtain $c(\xi) = \hat{c}(\hat{\xi})$ with $\hat{\xi} = \frac{\partial x_i}{\partial \hat{x}_j} \xi_i$. According to Section 1 in [2] uniform ellipticity is described by the inequality $A^{-1} \|\Xi\|_2^{2m} \leq |L(x, \Xi)| \leq A \|\Xi\|_2^{2m}$ for all $\Xi \in \mathbb{R}^{n+1}$ and all $x \in \overline{\Omega}$, where $L(x, \Xi)$ is the characteristic determinant of the PDE-system. The determinant is invariant under the hemisphere transformation in the sense that $\hat{L}(\hat{x}, \hat{\xi}) = L(x, \xi)$. As the first derivatives of T_x and T_x^{-1} exist and as these maps are in x uniformly bounded with respect to the norm $\|\cdot\|_{C^{3,\phi}}$ there is a constant ω such that $\omega^{-1} \|\hat{\xi}\|_2 \leq \|\xi\|_2 \leq \omega \|\hat{\xi}\|_2$ for every $x \in \mathcal{A}$, $\xi \in \Sigma_{R(x)}$ and $\hat{\xi} = T_x(\xi)$. Confer Chapter I.6.2 of [18], too. Finally, this results in the uniform ellipticity of the transformed system with the new ellipticity constant $A\omega^{2m}$.

Applying the Schauder estimates to both cases of Neumann boundary and homogeneous Dirichlet conditions yields the inequality statement. A uniform choice of the constant C in (17) with respect to \mathcal{O} is justified by the previous analysis of the constants $d, \kappa, \Delta_{\partial\Omega}$ and A . The replacement of $\|u_k\|_{C^0(\Omega)}$ by $\int_{\Omega} |u_k| dx$ for $k = 1, 2, 3$ is a consequence of a uniform cone property of \mathcal{O} and of Lemma 6.5. \blacksquare

By means of the following theorem and the properties of C^k -admissible domains we can show compactness of $\mathcal{G} = \{(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$ where $u(\Omega)$ uniquely solves $\mathcal{P}(\Omega)$.

Theorem 6.7 *There is a positive constant C such that the solution $u \in V(\Omega(\alpha))$ of the previous theorem satisfies $\|u_j\|_{C^{3,\phi}(\Omega)} \leq C$ for $j = 1, 2, 3$, where C can be chosen uniformly with respect to \mathcal{O} and forces $f_i \in C^{1,\phi}(\overline{\Omega^{\text{ext}}})$, $g_i \in C^{2,\phi}(\partial\Omega(\alpha))$ for $i = 1, 2, 3$, $\phi \in (0, 1)$ whose corresponding norms are uniformly bounded by k_{fg} .*

Proof: The main part of the proof consists of showing that there is a constant C independent of $\Omega \in \mathcal{O}$ such that $\|u\|_{H^1(\Omega(\alpha))} \leq C$. We follow ideas of the proof of Lemma 2.24 in [20]:

The weak formulation (14) of $\mathcal{P}(\alpha)$ can be rewritten in the form

$$\int_{\Omega} \sum_{i,j=1}^3 \sigma_{ij}(v) \varepsilon_{ij}(v) dx = \int_{\Omega} \sum_{i=1}^3 f_i v_i dx + \int_{\partial\Omega_N} \sum_{i=1}^3 g_i v_i dA$$

where $\sigma_{ij} = \sum_{k,l=1}^3 C_{ijkl} \varepsilon_{kl}$. The ellipticity constant $C_{ijkl} = \delta_{ij} \delta_{kl} \lambda + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ is given by the Lamé coefficients λ, μ and so a constant element of $C(\overline{\Omega^{\text{ext}}})$. Moreover, the constant satisfies the symmetries $C_{ijkl} = C_{jikl} = C_{klij}$ and there exists a constant $q > 0$ such that $C_{ijkl}(x) \xi_{ij} \xi_{kl} \geq q \xi_{ij} \xi_{kl}$ for all $x \in \overline{\Omega^{\text{ext}}}$. This results in

$$B_{\alpha}(v, v) = \int_{\Omega(\alpha)} \sum_{i,j=1}^3 \sigma_{ij}(v) \varepsilon_{ij}(v) dx \geq q \int_{\Omega(\alpha)} \sum_{i,j=1}^3 \varepsilon_{ij}(v) \varepsilon_{ij}(v) dx = q \|\varepsilon(v)\|_{[H^0(\Omega(\alpha))]^3}^2$$

for all $v \in V_{DN}$. Because of the assumptions for f and g

$$L_{\alpha}(u) = \int_{\Omega} \sum_{i=1}^3 f_i v_i dx + \int_{\partial\Omega_N} \sum_{i=1}^3 g_i v_i dA \leq C \|u\|_{[H^1(\Omega(\alpha))]^3}$$

holds and the constant is independent of $\alpha \in U^{ad}$. The previous inequalities and the weak equation (14) lead to $q\|\varepsilon(u)\|_{[H^0(\Omega(\alpha))]^3}^2 \leq C\|u\|_{[H^1(\Omega(\alpha))]^3}$. This and Korn's second inequality (15) imply $q\|\varepsilon(u)\|_{[H^0(\Omega(\alpha))]^3}^2 \leq C\|u\|_{[H^1(\Omega(\alpha))]^3} \leq C\|\varepsilon(u)\|_{[H^0(\Omega(\alpha))]^3}$, where C also depends on the constant of Korn's second inequality. As [29] shows, this constant is uniform with respect to a class of domains possessing a uniform cone property. Applying once more Korn's second inequality one obtains $\|u\|_{H^1(\Omega(\alpha))} \leq C$ with C independent of α .

This result, the inequality $\|v\|_{L^1(\Omega)} \leq \sqrt{\text{vol}(\Omega)}\|v\|_{L^2(\Omega)}$ for $v \in L^2(\Omega)$ and the results of the previous theorem show the statement of this corollary. \blacksquare

Lemma 6.8 *Let the setting of the previous theorem be given on an arbitrary sequence of domains $(\Omega(\alpha_n))_{n \in \mathbb{N}} \subset \mathcal{O}$. For $\phi \in (0, 1)$ and $i = 1, 2, 3$ let $f_i \in C^{1,\phi}(\overline{\Omega^{ext}})$ and $g_i \in C^{2,\phi}(\overline{\Omega^{ext}})$ such that $g_i|_{\partial\Omega(\alpha)_N} \in C^{2,\phi}(\partial\Omega(\alpha)_N)$ for $\alpha \in U^{ad}$ and such that their corresponding norms are uniformly bounded by a constant k_{fg} . Let $(\alpha_n, u_n)_{n \in \mathbb{N}}$ be a sequence of admissible shapes $\alpha_n \in U^{ad}$ and of the corresponding solutions $u_n \in V(\Omega(\alpha_n))$ of $\mathcal{P}(\alpha_n)$. Then, there exists a subsequence $(\alpha_{n_k}, u_{n_k})_{n_k \in \mathbb{N}}$ such that $\Omega(\alpha_{n_k}) \xrightarrow{\tilde{\mathcal{O}}} \Omega(\alpha)$ and $u_{n_k} \rightsquigarrow u$ as $n_k \rightarrow \infty$ for some $\alpha \in U^{ad}$ and for the corresponding solution $u \in V(\Omega(\alpha))$ of $\mathcal{P}(\alpha)$.*

Proof: Due to Lemma 5.7 there is a subsequence $(\alpha_{n_l})_{n_l \in \mathbb{N}}$ with $\Omega(\alpha_{n_l}) \xrightarrow{\tilde{\mathcal{O}}} \Omega(\alpha)$ as $n_l \rightarrow \infty$ for some $\alpha \in U^{ad}$. According to Theorem 6.6 $u_{n_l} \in [C^{3,\phi}(\overline{\Omega(\alpha_{n_l})})]^3$ holds for every $n_l \in \mathbb{N}$. Moreover, according to Theorem 6.7 there is a constant $C > 0$ independent of every $\Omega \in \mathcal{O}$ such that $\|u_{n_l}\|_{[C^{3,\phi}(\Omega(\alpha_{n_l}))]^3} \leq C$. Because of Lemma A.4 there is a constant C such that $u_{n_l}^{ext} = p_{\Omega(\alpha_{n_l})}u_{n_l}$ satisfies $\|u_{n_l}^{ext}\|_{[C^{3,\phi}(\Omega^{ext})]^3} \leq C\|u_{n_l}\|_{[C^{3,\phi}(\Omega(\alpha_{n_l}))]^3}$. Chapters I.6.2 and I.6.9 of [18] show that the constant can be chosen to be independent of the design variables α . The previous inequalities result in a uniform bound for all $\|u_{n_l}^{ext}\|_{[C^{3,\phi}(\Omega^{ext})]^3}$. As the unit ball in $C^{3,\phi}(\Omega^{ext})$ is compact in $C^3(\Omega^{ext})$ due to the Arzela-Ascoli theorem we find a further subsequence $(\alpha_{n_k}, u_{n_k})_{n_k \in \mathbb{N}}$ such that

$$\Omega(\alpha_{n_k}) \xrightarrow{\tilde{\mathcal{O}}} \Omega(\alpha), u_{n_k}^{ext} \xrightarrow{C^3} v \text{ as } n_k \rightarrow \infty \text{ for some } v \in [C^3(\Omega^{ext})]^3.$$

Because of $u_{n_k}^{ext} \xrightarrow{C^3} v$ as $n_k \rightarrow \infty$ the function $v|_{\overline{\Omega(\alpha)}}$ solves $\mathcal{P}(\alpha)$. According to Theorem 6.2 $\mathcal{P}(\alpha)$ has a unique solution $u \in V(\Omega(\alpha))$ and v is an extension u^{ext} of u . \blacksquare

As already mentioned we consider functionals which are volume or surface integrals with continuous integrands. In order to show continuity of the functionals we apply Lebesgue's dominated convergence theorem.

Lemma 6.9 *Let $\mathcal{F}_{vol}, \mathcal{F}_{sur} \in C^0(\mathbb{R}^{40}, \mathbb{R})$ and let the set \mathcal{O} only consist of C^0 -admissible shapes. For $\Omega \in \mathcal{O}$ and $u \in [C^3(\overline{\Omega})]^3$ with $q \geq 1$ and $\phi \in (0, 1)$ consider the volume integral*

$$J_{vol}(\Omega, u) = \int_{\Omega} \mathcal{F}_{vol}(x, u(x), \nabla u(x), \nabla^2 u(x)) dx.$$

Let $(\Omega_n)_{n \in \mathbb{N}} \subset \mathcal{O}$ with $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$ as $n \rightarrow \infty$ and let $(u_n)_{n \in \mathbb{N}}$ be a sequence with $u_n \in [C^3(\overline{\Omega_n})]^3$ and $u_n \rightsquigarrow u$ as $n \rightarrow \infty$ for some $u \in [C^3(\overline{\Omega})]^3$. Then, $J_{vol}(\Omega_n, u_n) \rightarrow J_{vol}(\Omega, u)$ as $n \rightarrow \infty$.

If the set \mathcal{O} only consists of C^1 -admissible shapes and if we consider a surface integral

$$J_{\text{sur}}(\Omega, u) = \int_{\partial\Omega} \mathcal{F}_{\text{sur}}(\cdot, u(\cdot), \nabla u(\cdot), \nabla^2 u(\cdot)) dA,$$

one obtains $J_{\text{sur}}(\Omega_n, u_n) \rightarrow J_{\text{sur}}(\Omega, u)$ as $n \rightarrow \infty$ as well.

Proof:

At first, we consider the volume integral. Using the characteristic function one obtains

$$J(\Omega_n, u_n) = \int_{\Omega^{\text{ext}}} \chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^{\text{ext}}(x), \nabla u_n^{\text{ext}}(x), \nabla^2 u_n^{\text{ext}}(x)) dx.$$

Because of $\mathcal{F}_{\text{vol}} \in C^0(\mathbb{R}^{40}, \mathbb{R})$ and $u_n \rightsquigarrow u$ as $n \rightarrow \infty$ there is a constant $C > 0$ such that the inequality $|\chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^{\text{ext}}(x), \nabla u_n^{\text{ext}}(x), \nabla^2 u_n^{\text{ext}}(x))| \leq C$ holds for all $n \in \mathbb{N}$ almost everywhere in Ω^{ext} . Moreover, $\Omega_n \xrightarrow{\mathcal{O}} \Omega$ and $u_n^{\text{ext}} \rightarrow u^{\text{ext}}$ in $[C_0^3(\Omega^{\text{ext}})]^3$ as $n \rightarrow \infty$ ensure the existence of

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^{\text{ext}}(x), \nabla u_n^{\text{ext}}(x), \nabla^2 u_n^{\text{ext}}(x))) \\ &= \chi_{\Omega}(x) \cdot \mathcal{F}_{\text{vol}}(x, u^{\text{ext}}(x), \nabla u^{\text{ext}}(x), \nabla^2 u^{\text{ext}}(x)) \end{aligned}$$

for all $x \in \Omega^{\text{ext}}$. Lebesgue's dominated convergence theorem can now be applied to permute integral and limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} J(\Omega_n, u_n) &= \lim_{n \rightarrow \infty} \int_{\Omega^{\text{ext}}} \chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^{\text{ext}}(x), \nabla u_n^{\text{ext}}(x), \nabla^2 u_n^{\text{ext}}(x)) dx \\ &= \int_{\Omega^{\text{ext}}} \lim_{n \rightarrow \infty} (\chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^{\text{ext}}(x), \nabla u_n^{\text{ext}}(x), \nabla^2 u_n^{\text{ext}}(x))) dx \\ &= \int_{\Omega} \mathcal{F}_{\text{vol}}(x, u(x), \nabla u(x), \nabla^2 u(x)) dx = J(\Omega, u). \end{aligned}$$

With respect to the surface integral similar arguments can be used and so we only address technical steps which are special to integrating over a surface. Because of their definition the boundary of every $\Omega \in \mathcal{O}$ is a differentiable submanifold and the surface integral

$$J(\Omega_n, u_n) = \int_{\partial\Omega_n} \mathcal{F}_{\text{sur}}(\cdot, u_n(\cdot), \nabla u_n(\cdot), \nabla^2 u_n(\cdot)) dA$$

is the sum of integrals over a finite number $|I|$ of map areas $\{\mathcal{A}_n^i\}_{i \in I}$. This number is the same for every $\Omega \in \mathcal{O}$ due to the definition of the basic design $\hat{\Omega}$ and of the design variables $\alpha \in U^{ad}$. The integrals have the form

$$\int_{\mathcal{A}_n^i} \mathcal{F}_{\text{sur}}(\varphi_n(s), u_n(\varphi_n(s)), \nabla u_n(\varphi_n(s)), \nabla^2 u_n(\varphi_n(s))) \sqrt{g^{\varphi_n}(s)} ds$$

where $\varphi_n^i : \mathcal{A}_n^i \rightarrow \mathbb{R}^3$ is a chart and $g^{\varphi_n}(s)$ the corresponding Gram determinant. The map $\varphi_n^\alpha : \overline{\Omega}_{2d} \rightarrow \mathbb{R}^3$, $\varphi_n^\alpha(s_1, s_2) = (s_1, s_2, \alpha_n(s_1, s_2))^T$, is the chart for the portion of the boundary $\partial\Omega_n$ which is described by the design variable¹⁴ $\alpha_n \in C^1(\overline{\Omega}_{2d})$. Because of $\mathcal{F}_{\text{sur}} \in C^0(\mathbb{R}^{40}, \mathbb{R})$ and $u_n \rightsquigarrow u$ as $n \rightarrow \infty$ and because of the fact that $g^{\varphi_n^\alpha}$ is bounded we can apply Lebesgue's dominated convergence theorem again to prove $J(\Omega_n, u_n) \rightarrow J(\Omega, u)$ as $n \rightarrow \infty$. ■

¹⁴This is the only part of the boundary which varies over the different domains of \mathcal{O} .

Now, we can prove the existence of optimal shapes and present our main result.

Theorem 6.10 (Existence of Optimal Shapes for SO Problems in Linear Elasticity)

Let $\mathcal{F}_{vol}, \mathcal{F}_{sur} \in C^0(\mathbb{R}^{40}, \mathbb{R})$ and recall $\tilde{\mathcal{O}}, \mathcal{O}$ of the previous definitions¹⁵. Consider the state problem $\mathcal{P}(\alpha)$ and the functional $J(\Omega, u) = J_{vol}(\Omega, u) + J_{sur}(\Omega, u)$ with

$$J_{vol}(\Omega, u) = \int_{\Omega} \mathcal{F}_{vol}(x, u(x), \nabla u(x), \nabla^2 u(x)) dx, \quad J_{sur}(\Omega, u) = \int_{\partial\Omega} \mathcal{F}_{sur}(\cdot, u(\cdot), \nabla u(\cdot), \nabla^2 u(\cdot)) dA$$

and $u = u(\Omega)$ unique solution of $\mathcal{P}(\alpha)$. With respect to $\mathcal{P}(\alpha)$ let the Lamé coefficients λ, μ be constant and for $\phi \in (0, 1)$ and $i = 1, 2, 3$ let $f_i \in C^{1, \phi}(\overline{\Omega^{ext}})$ and $g_i \in C^{2, \phi}(\overline{\Omega^{ext}})$ such that $g_i|_{\partial\Omega(\alpha)_N} \in C^{2, \phi}(\partial\Omega(\alpha)_N)$ for every $\alpha \in U^{ad}$. Moreover, let their corresponding norms be uniformly bounded by a constant k_{fg} independent of $\alpha \in U^{ad}$. Then, there exists an optimal shape $\Omega^* \in \mathcal{O}$ such that $J(\Omega^*, u(\Omega^*)) \leq J(\Omega, u(\Omega))$ for all $\Omega \in \mathcal{O}$.

Proof: Let $(\alpha_n, u_n)_{n \in \mathbb{N}}$ be a minimizing sequence of $\inf\{J(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$, where $u_n = u(\Omega_n) = u(\Omega(\alpha_n))$ is the unique solution of the state problem $\mathcal{P}(\alpha_n)$. Then, because of Lemma 6.8 there exists a subsequence $(\alpha_{n_k}, u_{n_k})_{n_k \in \mathbb{N}}$ such that $\Omega(\alpha_{n_k}) \xrightarrow{\tilde{\mathcal{O}}} \Omega(\alpha)$ and $u_{n_k} \rightharpoonup u$ as $n_k \rightarrow \infty$ for some $\alpha \in U^{ad}$ and for the corresponding solution $u \in V(\Omega(\alpha))$ of $\mathcal{P}(\alpha)$. Since all prerequisites of Lemma 6.9 are satisfied we obtain $J(\Omega_{n_k}, u_{n_k}) \rightarrow J(\Omega, u)$ as $n_k \rightarrow \infty$. Because $(\alpha_{n_k}, u_{n_k})_{n_k \in \mathbb{N}}$ is also a minimizing sequence of $\inf\{J(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$ the admissible shape $\Omega^* = \Omega(\alpha)$ is an optimal shape. ■

Next, we apply the result of the previous theorem to the cost functional (10), where now the number of cycles n is fixed but where the domains $\Omega = \Omega(\alpha) \in \mathcal{O}$ are variable.

Theorem 6.11 (Optimal Reliability)

Let the admissible shapes $\Omega = \Omega(\alpha)$ of \mathcal{O} be subject to surface driven LCF failure mechanism. Regarding $\mathcal{P}(\alpha)$ let the assumptions of Theorem 6.10 be imposed and let u be the unique solution of $\mathcal{P}(\alpha)$. Let N_{det} be the solution of the CMB equation (11) and consider the cost functional

$$J_{sur}(\Omega, u) = \int_{\partial\Omega} \left(\frac{1}{N_{det}(\varepsilon_a(\nabla u(x)))} \right)^m dA = \left\| \frac{1}{N_{det}(\varepsilon_a(\nabla u(\cdot)))} \right\|_{L^m(\partial\Omega)}^m, \quad (20)$$

where $x \mapsto \varepsilon_a(\nabla u(x))$ represents $x \mapsto RO \circ SD^{-1} \circ \frac{\sigma_v^e(\cdot)}{2} \circ \sigma^e(\cdot) \circ \nabla u(x)$. Then, there exists an optimal shape $\Omega^* \in \mathcal{O}$ such that $J_{sur}(\Omega^*, u(\Omega^*)) \leq J_{sur}(\Omega, u(\Omega))$ for all $\Omega \in \mathcal{O}$.

Proof: As all prerequisites of Theorem 6.10 are satisfied we directly obtain the existence of an optimal shape $\Omega^* \in \mathcal{O}$. ■

Theorem 6.11 proves that there is an optimal design $\Omega^* \in \mathcal{O}$ which minimizes the expected number of LCF crack initiations divided by n^m . This results in a minimum for the probability for LCF crack initiation and ensures optimal reliability. According to the remarks to Definition 4.1 volume driven HCF failure mechanism can be considered by volume integrals whose integrands only differ from the LCF integrands by the material parameters. In that case the existence of an optimal reliable design can be proven by means of Theorem 6.10 as

¹⁵ \mathcal{O} consists of C^4 -admissible shapes.

well. Moreover, note that a possible extension of our models to an inclusion of $\nabla^2 u$ and stress gradients can be considered via Theorem 6.10.

At the end of this section we prove the existence of an optimal design in case of the deterministic LCF model according to Definition 4.2. Note that the following result can also be extended to deterministic HCF.

Theorem 6.12 (*Deterministic LCF and Shape Optimization*)

Under the setting of Theorem 6.11 consider the functional

$$J_{sur}(\Omega, u) = \sup_{\Omega \in \mathcal{O}} \inf_{x \in \partial\Omega} N_{det}(\varepsilon_a(\nabla u(\Omega)(x))).$$

There exists an optimal shape $\Omega^ \in \mathcal{O}$ such that $J_{sur}(\Omega^*, u(\Omega^*)) \geq J_{sur}(\Omega, u(\Omega))$ for all $\Omega \in \mathcal{O}$.*

Proof: The proof of Theorem 6.10 yields all arguments needed except for the continuity property of J_{sur} . But this follows from uniform convergence required in Definition 6.3. \blacksquare

We have proven the existence of optimal designs for a general class of cost functionals without convexity constraints. The cases of our local and probabilistic model and of the deterministic model for LCF are included. Moreover, we have shown that these results can be extended to the case of volume driven HCF. In [19] we address sensitivity analysis and the existence of shape gradients in conjunction with our probabilistic model for fatigue.

A Appendix

Definition A.1 (*Hemisphere Property, Hemisphere Transformation*) [2]

Let $\mathcal{A} \subset \Omega$ be a subdomain such that $\partial\mathcal{A} \cap \partial\Omega$ is in the interior of Γ in the n -dimensional sense and let $d > 0$ be a positive constant. If every $x \in \mathcal{A}$ within a distance d of $\partial\Omega$ has a neighborhood U_x with $\overline{U}_x \cap \partial\Omega \subset \Gamma$, $B(x, d/2) \subset U_x$ and

$$\overline{U}_x \cap \overline{\Omega} = T_x(\Sigma_{R(x)}), \quad \overline{U}_x \cap \partial\Omega = T_x(F_{R(x)}), \quad 0 < R(x) \leq 1$$

for some hemisphere $\Sigma_{R(x)}$ and functions T_x, T_x^{-1} of some class $C^{k,\phi}$, \mathcal{A} is said to satisfy a hemisphere property. Moreover, the functions T_x are called hemisphere transformations.

Definition A.2 (*$C^{k,\phi}$ -Boundary, Lipschitz Boundary*) [18]

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $k \in \mathbb{N}$ and $0 \leq \phi \leq 1$. The subset Ω has a $C^{k,\phi}$ -boundary if at each point $x_0 \in \partial\Omega$ there exists $B = B(x_0, r)$ for some $r > 0$ and an injective $C^{k,\phi}$ -map $\psi : B \rightarrow D \subset \mathbb{R}^n$ such that $\psi(B \cap \Omega) \subset \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$, $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}_+^n$ and $\psi^{-1} \in C^{k,\phi}(D)$. If the maps ψ are only Lipschitz continuous Ω has a Lipschitz boundary.

Remark A.3 [18]

Let Ω be a bounded domain in \mathbb{R}^n and let $k \in \mathbb{N}$ and $0 \leq \phi \leq 1$. If every $x_0 \in \partial\Omega$ has a neighbourhood in which the boundary is locally described by a graph of a $C^{k,\phi}$ -function of $n-1$ of the coordinates x_1, \dots, x_n the domain Ω has a $C^{k,\phi}$ -boundary. Note that the converse is true if $k \geq 1$.

Lemma A.4 (*Extension Lemma*) [18], Part I, Section 6

Let $\Omega^{ext} \subset \mathbb{R}^n$ be open and let Ω be a $C^{k,\phi}$ -domain with $\overline{\Omega} \subset \Omega^{ext}$, $k \geq 1$ and $0 \leq \phi < 1$. For $\phi = 0$ it is $C^{k,0} = C^k$. If $u \in C^{k,\phi}(\overline{\Omega})$ there is a function $w \in C_0^{k,\phi}(\Omega^{ext})$ such that $w = u$ in $\overline{\Omega}$ and $\|w\|_{C^{k,\phi}(\Omega^{ext})} \leq C\|u\|_{C^{k,\phi}(\Omega)}$ for a constant C depending only on k, Ω and Ω^{ext} .

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